Mathematical theory of the waterbed

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(Received 24 February 1978; accepted 24 April 1978)

The support of a loading body by a waterbed involves a rather complicated interplay between tension (in an inextensible membrane) and buoyancy forces (in a fluid). In the idealization of a slab symmetry, the equilibrium equations for the shape of the membrane can be solved to give a two-parameter family of solutions. Mathematically, these parameters are (i) a dimensionless measure of the degree to which tension T is important, and (ii) a dimensional length $(T/g\rho)^{1/2}$ which sets the scale of the problem. The two parameters can be shown to be physically equivalent to (i) the weight of the loading body and (ii) the fullness of the waterbed. The problem of N bodies is also considered.

I. INTRODUCTION

A waterbed consists of a shallow rectangular trough ("frame") which contains a sealed, flat, flexible bag of water ("mattress"). Typical dimensions of frame and mattress are $1.8 \times 2.2 \times 0.25$ m. When there is no load on the waterbed, the mattress is more or less unstressed; it contains a volume of water sufficient to fill the frame. The water is supported below and on the sides by pressure on the frame, transmitted through the mattress. The upper surface of the water is approximately free, albeit covered by the mattress's flexible membrane.

Under load, the role of the mattress is more complicated, and two further properties of the membrane become significant: (i) although flexible, *the mattress is inextensible*, so that the total surface area of the water mass is conserved (as is its volume); (ii) the mattress is nowhere attached to the frame, so that reduction of the frame's normal force to zero at any point will result in the membrane's pulling away from the frame and becoming a new free surface (generally with nonzero stresses in the membrane).

In advertisements, objects are said to "float" on a waterbed, and the bed's support of a load is tacitly identified with the buoyancy force that would support the same load absent the mattress. In fact, the situation is somewhat more complex than just flotation. (For example, if the membrane were tough enough, an infinitely dense object would be supported after sinking only a finite distance into the bed and displacing only a finite volume of water.) Qualitatively, the response of the waterbed to a significant load (with volume ~0.1 that of the mattress, say, and density about the same as water) is as follows: The loading object sinks into the mattress by a fairly small fraction of its height. The water displaced by the object and by the smooth "well" around it forms a bulge in the mattress whose maximum is located between the object and the frame in all directions. The bulge pulls the side of the mattress away from the frame down to some finite depth. The mattress surface is under roughly constant tension everywhere above this level of contact.

There are three distinguishable contributions to the equilibrium upward force on the load, and they can all be of comparable magnitude: (i) force transmitted to the load directly by the tension of the mattress and its curvature around the load; (ii) hydrostatic pressure exerted on the load through the mattress from a water column rising to the highest point on the bulge; and (iii) hydrostatic pressure from an additional "fictitious" head of water which rises above the bulge due to the pressure transmitted by the tension of the mattress to the upper surface of the water.

The first contribution is distinguished from the other two by the fact that it concentrates at points of highest curvature on the load. For example, it acts only on the edges of any object with a flat lower surface. The second two contributions are proportional to area and increase with depth. They are both pressure forces, but are distinguished by where they originate. The geometrical height of the bulge determines one contribution. If the topmost point of the bulge were punctured and a standpipe attached, water would rise in the pipe by the height of the other contribution, the fictitious head.

In typical use, the waterbed is loaded by bodies which are much longer than they are wide, oriented parallel to the longest dimension of the frame. Further, the cross-sectional shape of the load is only slowly varying along its length. These facts suggest that the idealization of an infinitely long waterbed, uniform along its length and uniformly loaded, should preserve most features of interest in the system. This expectation seems to be borne out by observation: the longitudinal curvature and tension in the mattress is generally much less than the transverse curvature and tension, so each transverse cross section is separately quite close to the two-dimensional equilibrium of the idealized case. In the remainder of this paper, we restrict ourselves to the two-dimensional idealization.

II. SHAPE OF THE MEMBRANE

The following parameters enter into the analysis (see Fig. 1): T, transverse tension in the membrane; $P(y)$, water pressure as a function of height; ρ, density of water in the mattress; g, acceleration of gravity; y_0, ordinate of fictitious head zero-pressure surface; y_1, ordinate of membrane maximum; y_2, ordinate of contact of membrane with frame; $a \equiv (T/g\rho)^{1/2}$, dimensional length scale; $\xi \equiv y/a$ (nondimensional ordinate); $\xi_0 \equiv y_0/a$ (nondimensional ordinate; similarly ξ_1, ξ_2); $s \equiv x/a$ (nondimensional abscissa); $\eta \equiv \xi_0 - \xi_2$; $\zeta \equiv \xi_0 - \xi$; $\chi \equiv \xi - \xi_2$. The basic equation for the shape of the membrane $y(x)$ is obtained by equating the pressure at depth y to the product of the membrane's tension and (negative of the) radius of curvature[1]:

$$P(0) - g\rho y = -Ty''/(1 + y'^2)^{3/2}. \quad (1)$$

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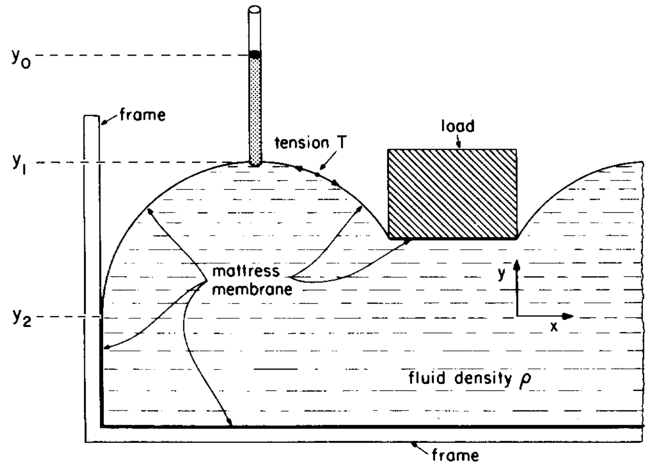

Fig. 1. Cross-sectional view of a loaded waterbed (schematic). A standpipe has been attached through the membrane to indicate the height of the "fictitious head" y_0 which is a measure of internal pressure.

Here $g\rho y$ is the weight of the water column from ordinate zero to y and prime denotes d/dx. Except for boundary conditions discussed in Sec. III, this equation is the only physics needed in the analysis; what remains is just mathematics.

If we substitute the dimensionless variables ξ and s for y and x, respectively, (1) becomes

$$\xi_0 - \xi = -\xi''(1 + \xi'^2)^{-3/2}, \qquad (2)$$

where

$$\xi_0 \equiv P(0)/(g\rho T)^{1/2}. \qquad (3)$$

Prime now denotes d/ds. The identification of ξ_0 as the height of the fictitious head follows from the fact that the pressure $P(0) - g\rho y$ vanishes when $\xi - \xi_0$ vanishes (i.e., this would be the free surface of fluid in a standpipe). Since Eq. (2) does not contain s, it can be integrated once by the trick of letting ξ' be a new independent variable. The result contains a new constant of integration, denoted ξ_2, and can be written

$$\xi' = \left(\frac{4}{[(\xi_2 - \xi_0)^2 - (\xi - \xi_0)^2]^2} - 1\right)^{1/2}. \qquad (4)$$

At $\xi = \xi_2$, ξ' becomes infinite and the membrane has a vertical tangent. That the point of contact with the side of the frame must occur at this point can be seen as follows: Below the contact point the membrane is pressed against the side. If the contact occurred at any noninfinite value of ξ', then the discontinuity in ξ' would imply a δ function in ξ'' and in the radius of curvature; this could not possibly be in equilibrium with a finite pressure. In actuality, the membrane would pull away from the side starting at this point and continuing down to some greater depth where it could be vertically tangent.

The maximum height of the membrane occurs where $\xi' = 0$. By Eq. (4) this occurs at

$$\xi_1 = \xi_0 - [(\xi_0 - \xi_2)^2 - 2]^{1/2}. \qquad (5)$$

In Eq. (5), the choice of the first minus sign is uniquely dictated by the condition that fluid pressure cannot be negative at the maximum (i.e., $\xi_0 - \xi$ must be positive). The choice of the third minus sign is necessary for a maximum (as opposed to a minimum; see Sec. V).

To proceed further, we need to integrate Eq. (4), which depends on a single parameter $\eta = \xi_0 - \xi_2$ and can be rewritten either

$$\zeta' = -[4\eta^{-4}(1 - \zeta^2/\eta^2)^{-2} - 1]^{1/2} \qquad (6a)$$

or

$$\chi' = [4\chi^{-2}(\chi - 2\eta)^{-2} - 1]^{1/2}. \qquad (6b)$$

Equation (6a) can actually be integrated to give a complicated closed-form expression involving elliptic integrals, but this is not in itself very enlightening. An alternative strategy is to examine the limiting cases of Eq. (6) as the parameter η is varied.

A. Case of $\eta \gg 2$; the highly stressed waterbed

If $\eta \gg 2$ then, from Eq. (5),

$$\xi_1 - \xi_2 = \eta - (\eta^2 - 2)^{1/2} \approx 1/\eta \ll \eta \qquad (7)$$

so the maximum relief of the membrane from frame contact to bulge maximum is negligible in comparison with the height of the fictitious head. Equation (6b) is now (to lowest order in $1/\eta$)

$$\chi' = [\chi^{-2}\eta^{-2} - 1]^{1,2}, \qquad (8)$$

which is easily integrable, giving

$$\eta^2\chi^2 + (\eta s - 1)^2 = 1, \qquad (9)$$

which is a circle of radius $1/\eta$. (The constant of integration was chosen to make $s = 0$ at the point of contact $\chi = 0$.) The physical meaning of this limit is that it corresponds to a bulge so overfull of fluid that it takes on (in the limit) that shape which maximizes its area at constant perimeter, the circle. Below, we will examine the conditions of loading which result in this case.

B. Case of $\eta \approx 2$; the lightly stressed waterbed

As η is decreased from the large values of the previous case, the height of the maximum ξ_1 starts to approach the zero-pressure surface ξ_0. Naive extrapolation of the solution (9) would indicate that ξ_1 reaches ξ_0 when $\eta \sim 1/\eta$, i.e., at $\eta \sim 1$. Actually, the exact value, $\eta^2 = 2$, can be read off from Eq. (5). We are therefore led to examine the behavior of the solution in a neighborhood of this value.

First, consider $\eta^2 < 2$. In this case, according to Eq. (4), ξ' is still positive at $\xi = \xi_0$. Above ξ_0, there can be no fluid (since the surface of zero pressure must be a free fluid surface). The case $\eta^2 < 2$ is therefore not relevant to the theory of the waterbed at all; it describes instead something like the side bulge of a vinyl-lined swimming pool whose flexible liner is attached to a frame above the water surface.

Next, consider η^2 slightly greater than 2. We can conveniently parametrize the excess of η^2 over 2 by a positive constant ϵ,

$$4/\eta^4 \equiv 1 - \epsilon. \qquad (10)$$

In this case, Eq. (6a) becomes (to lowest order in ϵ and lowest order in ζ/η)

$$\zeta' = (\zeta^2 - \epsilon)^{1/2}, \qquad (11)$$

whose solution is

$$\zeta = \epsilon^{1/2}\cosh(s_0 - s), \qquad (12)$$

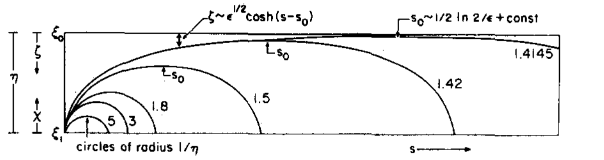

Fig. 2. Solutions to the waterbed equation, Eq. (6). The curves are labeled by their dimensionless parameter η. The vertical coordinate of each curve is scaled to make $\chi = \eta$ at the top of the graph.

where s_0 (a constant of integration) is the abscissa of the maximum ξ_1. As the membrane maximum approaches the zero-pressure surface, in other words, it flattens from a circle to a cosh function. The location of the maximum s_0 also moves to larger values of s as ϵ is decreased. As a first crude estimate, one can simply extrapolate the cosh function beyond its region of strict validity, setting (at $s = 0$)

$$\eta \approx \sqrt{2} \sim \epsilon^{1/2} \cosh(s_0), \tag{13}$$

which implies

$$s_0 \sim (1/2) \ln(2/\epsilon). \tag{14}$$

However, it is not difficult to give a more accurate treatment: If ϵ is small, then there is some range of ζ such that

$$\epsilon^{1/2} \ll \zeta \ll \eta. \tag{15}$$

This region is the overlap region of validity of two different limiting approximations. When ζ is in the range of inequality (15) *or smaller*, then Eqs. (11) and (12) are valid. When ζ is in the range of Eq. (15) *or larger*, then Eq. (6a) has a nontrivial lowest order which does not involve ϵ at all. Setting $\eta^2 = 2$ in Eq. (6a) gives

$$\zeta' = [(1 - \zeta^2/2)^{-2} - 1]^{1/2}, \tag{16}$$

which can be integrated[2] and simplified to

$$s = \cosh^{-1}(2/\zeta) + 2(1 - \zeta^2/4)^{1/2} - \sqrt{2} - \cosh^{-1}\sqrt{2} \quad (\zeta \gg \epsilon^{1/2}). \tag{17}$$

The constant of integration has been chosen to make $\zeta = \eta = \sqrt{2}$ at $s = 0$.

Now we can compare Eqs. (17) and (12) in their region of overlap [Eq. (15)] and (using $\cosh^{-1}x \approx \ln 2x$) easily read off an accurate value for s_0,

$$s_0 = (1/2)\ln(2/\epsilon) + (5/2)\ln 2 - \sqrt{2} - \cosh^{-1}\sqrt{2}, \tag{18}$$

which differs from our crude estimate [Eq. (14)] only by a constant. Equations (10), (12), (17), and (18) describe completely the case $\eta \approx 2$. That this case corresponds to a lightly stressed waterbed is seen in the fact that the cosh functions [Eqs. (12) and (17)] give a very flat upper surface to the membrane, implying that the surface is very nearly at zero pressure (has little pressure exerted on it by tension in the mattress).

Figure 2 plots the functions ξ (or ζ) as a function of s, for various values of η. For these curves the exact Eq. (4) has been integrated numerically. The asymptotic limits that we have derived here are indicated on the figure.

III. RESPONSE TO A CENTERED LOAD

We can now see what happens when a load is placed in the middle of the waterbed. The width of the load and bed are fixed. There is a two-parameter family of possible membrane configurations: First, the shape of the membrane depends on η; second the relative scale for this shape depends on $a \equiv (T/g\rho)^{1/2}$. Neither η nor a is known *a priori*. In Fig. 3, therefore, we draw representative configurations from the entire a, η plane, for the case of a rectangular load $^1/_4$ the width of the waterbed. To decide which configuration is the "correct" one, we must impose three physical constraints: First, the perimeter of the mattress (including its shape around the load) must be correct. This constraint can be used to establish the vertical position of the lower (flat) surface which is in contact with the bottom of the frame. (This surface is not shown in the figure.) Second, knowing the position of the lower surface, we must choose a configuration with the correct total volume (which

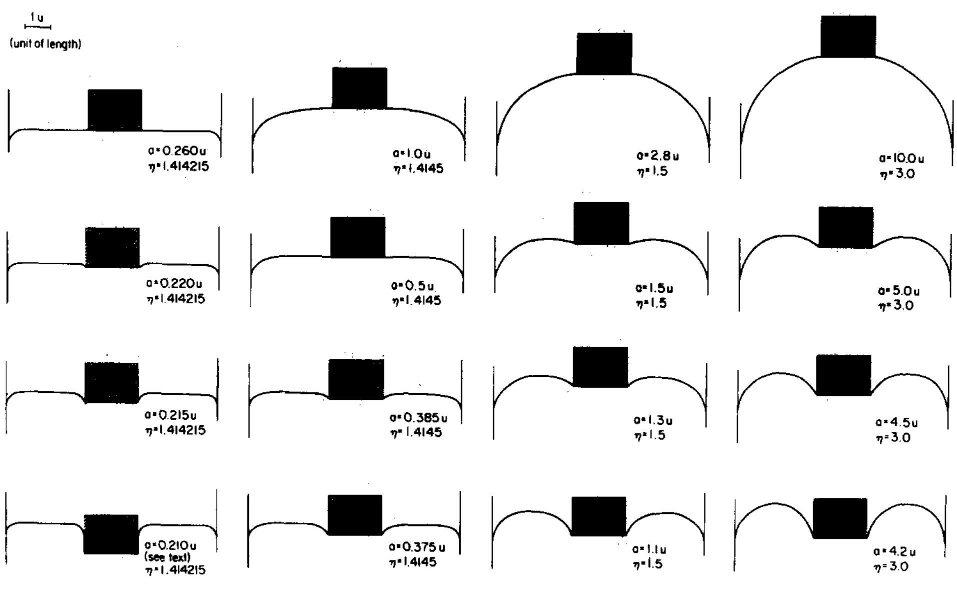

Fig. 3. The two-parameter family of solutions for a load of given size and shape. The weight of the load and the fullness of the bed vary over the plane of solutions. See text for details.

depends on how full the bed is). Third, the sum of upward forces on the body must equal its weight. These latter two physical parameters, the fullness of the waterbed and the weight of the object, are thus equivalent to the mathematical parameters a and η. Lines of constant fullness or weight are not lines of constant a or η, however. In the figure, the direction of increased load at constant volume is roughly from upper left to lower right, while the direction of fuller mattress at constant load is roughly from lower left to upper right. The extreme-upper-right configuration is that of a mattress filled to such an extreme overpressure that it bulges into a semicircle. Other figures at the extreme right correspond to extreme overpressures induced by very heavy objects placed on beds of varying fullness; the membrane surfaces are close to segments of circles, and the object is supported by tension forces and fictitious-head pressures of comparable magnitude. In the opposite limit, the configurations on the extreme left are supported exclusively by the buoyancy force of the geometrical head, corresponding to floating objects on a very underfilled mattress. When a is such as to make the distance from object to frame exactly $2s_0$, then a is a bad "coordinate" for the plane of solutions, since it does not define a unique vertical height for the object. Any vertical height does define a unique a, however. The lower-left example in the figure illustrates this degeneracy, which could also have been similarly shown at the bottom of each column of figures.

To find the correct solution for some particular finite weight and fullness one proceeds as follows: Apply the perimeter constraint and draw a family of configurations in the a, η plane. Next, apply the volume constraint to narrow down the family to a single parameter. Consider each member of the sequence in turn: from g, ρ, and a, compute $T = a^2 g\rho$, the tension. From this and the bend of the membrane around the load, compute the upward tension force. From its position of contact with the side (ξ_2), η, and a, compute $y_2 = a(\xi_2 + \eta)$, the height of the fictitious head. Using this head, compute the upward pressure force. The unique solution is that one which matches the sum of pressure and tension forces to the weight of the object. The kind of search described here is readily implemented as an iterative numerical procedure.

IV. GENERAL EQUATIONS FOR MANY BODIES

Figure 4 shows a more general case with several objects at specified horizontal positions. We want to see that the solutions already derived are sufficient to treat this case,

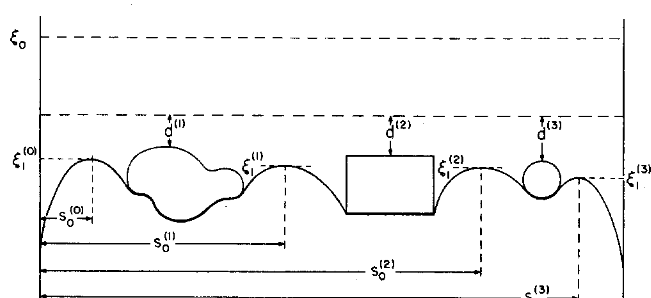

Fig. 4. Schematic of the "N-body problem" showing variables that enter the analysis.

i.e., that we have as many equations as unknowns. Suppose that there are N objects. Then the unknown quantities are $d^{(1)}, \ldots, d^{(N)}$, the heights of the objects; $s_0^{(0)}, \ldots, s_0^{(N)}$, the abcissas of the $N+1$ bulge maxima; $\xi_1^{(0)}, \ldots, \xi_1^{(N)}$, the ordinates of these maxima; $\eta^{(0)}, \ldots, \eta^{(N)}$, the shape parameters for each bulge; T, the mattress tension (which with g and ρ gives a); and ξ_0, the height of the fictitious head. There are $4N + 5$ unknowns in all.

The equations to be satisfied are: weight balance on N objects; tangency of the left and right sides of each bulge to its corresponding object or to the side of the frame [$2(N+1)$ conditions]; Eq. (5) applied to each bulge with the single value of ξ_0 but individual values of $\xi_1^{(i)}$ and $\xi_0 - \xi_2^{(i)} = \eta^{(i)}$ ($N + 1$ conditions); volume of mattress correct; and perimeter of mattress correct. These total $4N + 5$ equations, and allow, generically, a unique solution of the problem.

Note that we are not able to demand horizontal force balance or torque balance on the objects. There are not enough unknowns to support these extra conditions. In general, there will be a tendency for bodies on a waterbed to slide and/or roll horizontally in one direction or another. (In practice, friction keeps objects from sliding, and a slight inequality in the tension of different bulges brings the horizontal force into balance; for bodies other than circular, a similar small readjustment of angle brings them into torque balance; circular bodies do, in fact, roll on a waterbed.) An important unsolved problem in waterbed theory is the question for two bodies: for what range of mattress fullness and separation do flat bodies tend to slide toward each other? (Experimentally, bodies tend to separate when they start far apart, and they tend to attract when they both start close to the middle; but this behavior is evidently a function of mattress fullness as well.)

V. GENERALIZATIONS AND UNSOLVED PROBLEMS

We have assumed above that when $\xi(s)$ becomes tangent to a vertical line (at $\xi = \xi_2$) it joins onto a side of the frame. Another mathematical possibility is that it bends under to form the free, curved lower surface of a flexible water mattress which has been placed directly on the floor (rather than in a frame). In this case, Eq. (4) continues to hold, with the square root on the left-hand side changing sign at $\xi = \xi_2$. The boundary condition of a flat floor is now imposed at the point where the membrane becomes horizontal. $\xi' = 0$, which occurs at some ξ_3 [compare Eq. (5)].

$$\xi_3 = \xi_0 - [(\xi_0 - \xi_2)^2 + 2]^{1/2}. \tag{19}$$

It is the third sign which determines whether $\xi' = 0$ is a maximum (top surface) or minimum (bottom surface). One can explore lower-surface solutions analogously to our exploration of the upper surface in Sec. II.

There are several practical disadvantages to a waterbed without a frame: (i) The mattress is subject to higher stresses (there is no limit analogous to a load floating on a loose mattress). (ii) The mattress as a whole tends to "walk" when the load is unsymmetric; there is a net unbalanced horizontal force on the mattress. It is an interesting exercise to work out these effects quantitatively.

A most basic unsolved problem is: How full should a waterbed be? Too full, and the tendency is for the load to slide off. Too empty, and the membrane climbs unpleasantly up the side of the load. Suppose that "perfect fullness" is to have the membrane vertically tangent at just the bottom edge of each flat body. Can this condition be achieved simultaneously for more than one body? For bodies of different sizes or weights? At different separations?

In the limit of $\rho \rightarrow 0$, our theory reduces to the theory of the air mattress. It is not hard to see that the membrane is always circular in this case ($\eta \gg 2$). An air mattress is not nearly as comfortable as a waterbed. Why? A "mercury-bed" where ρ becomes large also seems unpromising. Evidently, for comfort, ρ should be close to the density of the body on the bed. Is this easily explained?

ACKNOWLEDGMENT

The author thanks the Alfred P. Sloan Foundation for financial support.

[1]The formula for radius of curvature, $y''/(1 + y'^2)^{3/2}$, is derived in most elementary calculus texts, e.g., M. Kline, *Calculus*, (Wiley, New York, 1967), Vol. 1, p. 458; or G. B. Thomas, Jr., *Calculus and Analytic Geometry*, 3rd ed., (Addison-Wesley, Reading, MA, 1960), p. 589.

[2]H. B. Dwight, *Tables of Integrals and other Mathematical Data* (Macmillan, New York, 1961), Secs. 341.01 and 321.01.