When to Bail from an Escalating Expense

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Abstract

They keep raising the rent on your self-storage unit while advertising low introductory prices for identical units. What should you do? (Hint: The answer involves Lambert's W-function.)

1 Statement of the Problem

Like a million or so others, I rent a self-storage unit from a well-known large firm at one of their thousands of locations in the U.S. The company's business model seems to be this:

- Offer new customers a very low monthly rent.
- Raise the rent annually by $\sim 15\%$.
- Continue raising the rent until the customer realizes that the cost has become exhorbitant and moves out.
- Go back to step one.

When the customer decides to move out, there are often identical empty units being advertised at a much lower cost (see first bullet above). You might think that the customer could do a deal with the location manager—move out only on paper, get the lower rent for the "vacant" unit, and pretend to move in again. But no, the company is on to this. My location manager explained that a vacated unit doesn't go on the market again until a district manager verifies that it is actually empty, and possibly even clean.

What I can do, however, is:

- Rent a new, second unit at the advertised low price.
- Move all my junk from the first to the second unit.
- Give up my first unit.

This costs. I have to pay for a full month of overlap with both units. I have a lot of junk. I mean *a lot!* I have to pay an hourly moving team for at least two hours of work. Plus the inconvenience, which, for the purposes of this note, I imagine translating into a dollar cost.

The question is, if I want to minimize my average monthly cost over the long run, how often should I do the moving-out-moving-in process, hereafter termed "bailing"? This is a problem also met in other contexts. For example, as the annual repair costs on your car start to escalate, when should you buy a new car? I call this class of problems "When to Bail?" The storage unit variant is a nicely pure example of this class, because (for the rules given above) we know in advance how the cost will increase, and we assume that the transition cost is known.

2 Solution

Suppose that the initial rent is R_0. Although, in real life, the rent increases by discrete multiplicative steps annually, we will instead solve the continuum analog, where the rent increases expoentially,

$$R(t) = R_0 \exp(\lambda t) \tag{1}$$

where R_0 is the initial (low) rent, and λ is the monthly rate of increase. For 15% annual increase, we have $\lambda = 0.15/12$, for example. If C_0 is the transition cost, and T_0 is the bail time, then the total cost of one complete cycle is

$$C(T_0) = C_0 + \int_0^{T_0} R_0 \exp(\lambda t) dt \tag{2}$$

The average cost per month of one full cycle is (doing the integral)

$$R_1(T_0) = C(T_0)/T_0 = \frac{C_0}{T_0} + \frac{R_0}{\lambda T_0}[\exp(\lambda T_0) - 1] \tag{3}$$

We minimize this by setting the derivative of R_1 with respect to T_0 to zero,

$$0 = \frac{dR_1}{dT_0} = \frac{R_0}{T_0} \left[\frac{1}{\lambda T_0} + \exp(\lambda T_0) \left(1 - \frac{1}{\lambda T_0} \right) - \frac{C_0}{R_0 T_0} \right] \tag{4}$$

This equation cannot be solved for T_0 in terms of elementary functions or the more common special functions, but it can be solved in terms of "Lambert's W-function" $W(z)$ (known to Mathematica as `ProductLog`), defined as the inverse function to $W \exp(W)$. Explicitly, $W(z)$ satisfies,

$$z = W(z) \exp[W(z)] \tag{5}$$

In terms of the Lambert W-function, equation (4) can be solved as,

$$y \equiv \lambda T_0 = 1 + W \left[\frac{1}{e} \left(\frac{\lambda C_0}{R_0} - 1 \right) \right] \equiv 1 + W \left[\frac{1}{e} (x - 1) \right] \tag{6}$$

where the obvious nondimensional quantities are here defined as $y \equiv \lambda T_0$ and $x \equiv \lambda C_0 / R_0$. The quantity y can be interpreted as the optimal number of e-folds of rent increase before bailing. C_0/R_0 is the number of months it would take (at the initial rent R_0) to equal the cost of bailing. So x can be interpreted as the number of e-folds of rent increase in that time.

The function $y(x)$ is plotted in Figure 1.

As a numerical example, my unit's initial rent is $R_0 = \$170$, now increasing at 15% annually. I estimate the cost of switching units as $C_0 = \$1000$ (mostly my inconvenience cost). Then

$$\lambda = 0.0125, \quad x = \lambda C_0 / R_0 = 0.0735 \tag{7}$$

which gives (via Mathematica or Figure 1),

$$y = 0.342, \quad T_0 = y/\lambda = 27.3 \text{ (months)} \tag{8}$$

So, I should optimally switch units a little over every two years. If that seems too often, the implication is that I am underestimating my inconvenience cost. Increasing C_0 to \$2000 yields $y/\lambda = 37.0$ months, or about every three years.

Actually, one should increase the estimates (in months) by about 6 months, because the rent is increased not continuously (as the model assumes), but at the end of a year.

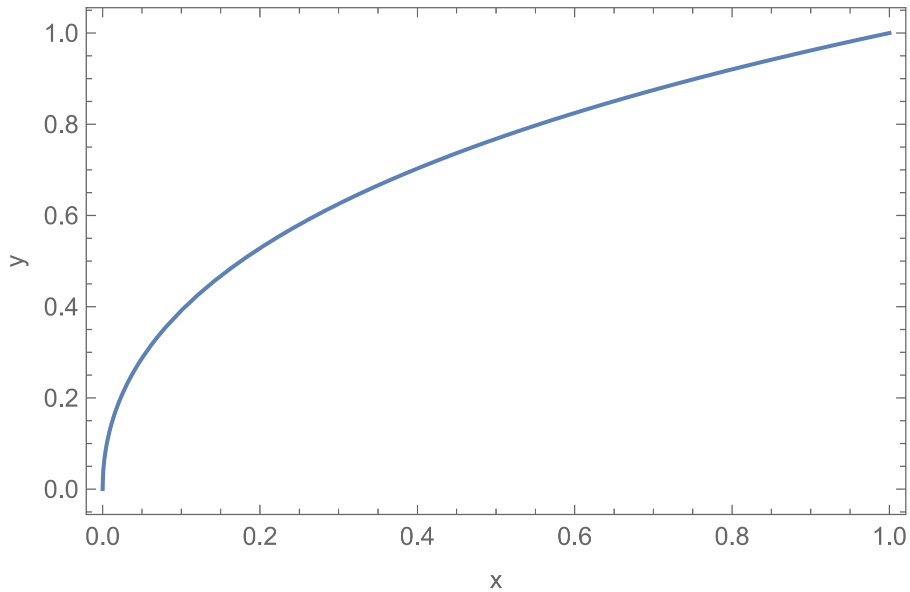

Figure 1: Equation (6) for the function $y(x)$ as numerically evaluated.

3 Rational Approximation

Evaluating the Lambert W-function outside of Mathematica (or another such package) is inconvenient. As an alternative, in the range $0 < x < 1$, equation (6) is well approximated by the rational approximation

$$y_{\mathrm{approx}} = \frac{\sqrt{x}\,(0.169057\sqrt{x} + 1.41358)}{1 + 0.582357\sqrt{x}} \tag{9}$$

Figure 2 plots the difference $y - y_{\mathrm{approx}}$, showing that it remains small in the range $0 < x < 1$.

4 Discussion

Wikipedia tells us that Lambert's W-function was first described in the form of equation (5) by Euler in 1783. The reason that it is sometimes called the productlog function is by analogy: $\log(x)$ the inverse function $\exp(x)$, so $\mathrm{productlog}(x)$ is taken as the inverse function to $x\log(x)$.

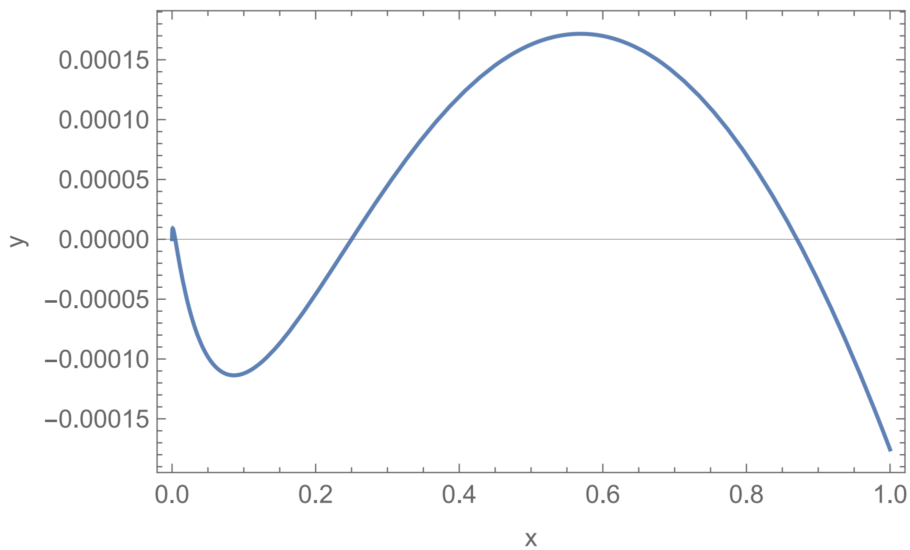

Figure 2: Difference $y - y_{\mathrm{approx}}$, cf. equations (6) and (9).