Did Dürer Intentionally Show Only His Second-Best Magic Square?

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1 Introduction

Albrecht Dürer's magic square, which appears as one of many scientific-mathematical allusions in the 1514 engraving *Melencolia I* [1], plays a key role in Dan Brown's recent popular thriller, *The Lost Symbol* [2]. No longer a mere artifact of recreational mathematics, the Dürer square has been elevated to the status of pop cultural meme. The square, which we will call "DA" (for reasons soon to be clear) is

$$\begin{bmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{bmatrix} \equiv \mathrm{DA} \tag{1}$$

DA is of order $n = 4$ and is "normal," meaning that it is composed of the positive integers 1 to n^2, and that its rows, columns, and diagonals sum to

$$S = \frac{n(n^2+1)}{2} = 34 \tag{2}$$

Within DA, the sum S is also to be found in a variety of other places [3] (hereafter abbreviated by the terms following in italics), including,

- the four 2×2 quadrants, e.g., $16 + 3 + 5 + 10$ (*quads*)
- the outer corners, $16 + 13 + 4 + 1$ (*corners*)
- the inner 2×2 box, $10 + 11 + 6 + 7$ (*inner*),
- the corners of the four enclosed 3×3 squares, e.g., $16+2+9+7$ (*3-squares*),
- the corners of the centered 4×2 and 2×4 rectangles $3 + 2 + 15 + 14$ and $5 + 8 + 9 + 12$ (*rects*)
- the corners of the two diagonal 2×3 rectangles $2 + 8 + 15 + 9$ and $5 + 3 + 12 + 14$ (*diagrects*)

- the two skewed squares $8+14+9+3$ and $2+12+15+5$ (*skewsquares*)
- the cruciform (Latin cross) shapes $3+5+15+11$ and $2+10+14+8$ (*latin*)
- the upside-down cruciforms (St. Peter's cross) shapes $3+9+15+7$ and $2+6+14+12$ (*peter*)

AD also has an additional symmetry,

- any two elements that are symmetric around the center sum to 17 (*sumcenter*)

The *sumcenter* property is sufficient to imply several of the properties already mentioned, namely *corners*, *inner*, *rects*, *diagrects*, and *skewsquares*; but it is not a necessary condition for all of these, as will become clear.

Not to be overlooked are the artist' s initials, DA, in the lower corners, with entries representing the fourth and first letters of the alphabet; and the year of creation, 1514, centered in the lower row. Noting that Dürer signed or titled his compositions variously at both top and bottom edges, it seems clear that the initials and year along the top row would also have met the artists needs. Similarly likely is that "A 1514 D" would have been as good as "D 1514 A". Arguably it would have been not just as good, but better, since Dürer's monogram (larger capital A enclosing smaller capital D) is surely intended to be read as "A D," not as "D A". For these reasons, we will define the property *1514AD* as being satisfied by any of the four cases, top or bottom, AD or DA.

The question we ask is, just how magical is DA? Are there many squares with all of the above properties, so that Dürer need only to have stumbled on one such? Or is DA the unique magic square with the above (or closely equivalent) properties?

The answer turns out to be somewhat interesting. DA is one of exactly two squares in a natural equivalence class of squares sharing all the above properties. And, it is the *inferior* one of the two, in the sense that the other class member, which we will call DA′, has an additional symmetry that DA lacks. Furthermore, we will see that it is very plausible that Dürer knew of the existence of DA′, and that his choice of showing DA and not DA′ was therefore likely intentional.

2 The Hierarchy of "Magical" Properties

Following Kraitchik's magisterial treatment [4], let us label the elements of the square as

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & p & q \end{bmatrix} \tag{3}$$

2.1 Normal Properties (Least Magical)

Requiring that the rows, columns, and diagonals sum to S adds $4+4+2-1=9$ constraints (subtracting the 1 because the sum of sums of rows and sum of sums of columns must be the same), leaving $16-9=7$ free parameters. Kraitchik [4] writes this most general "normal" magic square as

$$\begin{bmatrix} a & b & c & -a-b-c+S \\ e & f & g & -e-f-g+S \\ i & \begin{matrix} 2a+b+c+e \\ -g+i-S \end{matrix} & \begin{matrix} -2a-b-c-e \\ -f-i+2S \end{matrix} & f+g-i \\ \begin{matrix} -a-e \\ -i+S \end{matrix} & \begin{matrix} -2a-2b-c-e \\ -f+g-i+2S \end{matrix} & \begin{matrix} 2a+b+e+f \\ -g+i-S \end{matrix} & \begin{matrix} a+b+c \\ +e+i-S \end{matrix} \end{bmatrix} \tag{4}$$

where the free parameters are a, b, c, e, f, g, i.

Evidently, not all substitutions of integers in the range $1 \ldots 16$ for the free parameters yield valid magic squares. One must check that the formulas in (4) yield integers in the range $[1, 16]$ and (much more restrictive) yield each unused integer exactly once. By modern standards of computation, the number of trial hypotheses, $16 \times 15 \times \cdots \times 10 = 57657600$, is small and easy to check exhaustively in less than a second. The well-known answer is that there are 7040 distinct magic squares. If we count as equivalent squares that are related by the 8-element dihedral group D_4 (rotations and reflections of a square), then the number of nonequivalent squares is $7040/8 = 880$.

The magic square (4) is most general in the sense that it is required to satisfy only the rows, columns, and diagonals property. However, it is easy to check that, in the above parameterization, *corners*, *inner*, and *rects* also hold. So these get no extra credit and are also normal, "least magical," properties.

2.2 Algebraic Properties (More Magical)

We might now wish to add the magic of an additional property, perhaps *quads*, *3-squares*, or *diagrects*. But which? Luckily, we need not decide, since these are all equivalent! As can be checked from the relations in (4) (easily, using Mathematica), these properties all both imply and are implied by just one additional relation among the parameters, namely the upper left quadrant,

$$a+b+e+f=S \tag{5}$$

Kraitchik [4] terms these squares "algebraic".

2.3 Symmetric Sums Property (Most Magical)

What if we had wanted to impose *sumcenter* before, or instead of, the *quads*, etc. properties of §2.2? Requiring $h + i = S/2$ implies

$$i = e + f + g - S/2 \tag{6}$$

Next requiring $g + j = S/2$ implies

$$g = 2S - 2a - b - c - 2e - f \tag{7}$$

We now find that the relations $a + q = S/2$, $e + l = S/2$, $f + k = S/2$, and $d + m = S/2$ are all tautologically true, while the relationships $c + n = S/2$ and $b + p = S/2$ are each equivalent to $a + b + e + f = S$, which is just equation (5). So *sumcenter* implies *quads* (and its other equivalent relations), but not vice versa. Kraitchik [4] terms squares with these properties "algebraic central".

There is thus a natural hierarchy of "magicalness": A subset of the normal squares are algebraic (*quads*, etc.), and a subset of these are "central" (*sumcenter*).

2.4 A Plausible Construction Path

Dürer's square DA is algebraic central. One might think that it is therefore difficult to construct. However, the opposite is true. As the conditions on a square become more restrictive, it becomes easier to determine by trial and error whether a desirable one exists and, if so, to construct it. For example, if we imagine the artist *starting* with the idea of the *1514AD* property, then *sumcenter* makes the square almost trivial to construct: Start with either "1 15 14 4" or "4 15 14 1" in the bottom row. Use *sumcenter* to get the top row. Now the middle two rows must contain the integers $5, 6, \ldots, 12$. Try each in any fixed position in the inner quadrant; using *sumcenter*, rows, columns, and diagonals, and it becomes immediately clear that only one value will work. Finally, try each remaining integer in any remaining space; again only one will work. This construction produces the four squares

$$\begin{bmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{bmatrix} \equiv \mathrm{DA} \quad \begin{bmatrix} 16 & 3 & 2 & 13 \\ 9 & 6 & 7 & 12 \\ 5 & 10 & 11 & 8 \\ 4 & 15 & 14 & 1 \end{bmatrix} \equiv \mathrm{DA}'$$
$$\begin{bmatrix} 13 & 3 & 2 & 16 \\ 8 & 10 & 11 & 5 \\ 12 & 6 & 7 & 9 \\ 1 & 15 & 14 & 4 \end{bmatrix} \equiv \mathrm{AD} \quad \begin{bmatrix} 13 & 3 & 2 & 16 \\ 12 & 6 & 7 & 9 \\ 8 & 10 & 11 & 5 \\ 1 & 15 & 14 & 4 \end{bmatrix} \equiv \mathrm{AD}' \tag{8}$$

By construction, these (and their corresponding four reflections around a horizontal axis) are the *only* squares satisfying *sumcenter* and *1514AD*. If Dürer constructed DA by the above method, he surely would also have found the other three, here denoted DA′, AD, and AD′. Why did he pick DA as the one to be represented in *Melencolia I*?

3 The Cruciform Hierarchy

It is a matter for speculation as to whether Dürer recognized the Latin and St. Peter crosses that sum to S in DA. Given his time and place, and the multiple religious symbols already evident in *Melencolia I*, however, it is hard to believe that he did not. A further inferential bit of evidence is the fact that the squares AD and AD′, which give the artist's initials in their correct order, but which lack cruciform sums, were (if known to Dürer at all) rejected in favor of DA. So, let us look at the implications of cruciform sums as regards DA's uniqueness, or lack thereof.

Let us start with a normal square, not necessarily algebraic or central. If we impose the condition of a single Latin cross, $b + e + g + n = S$, we easily find that the second Latin cross $c + f + h + p = S$ is automatically satisfied, yielding the property we have called *latin*. Similarly, for St. Peter's crosses, because it is the same algebra on the upside down square, a single condition implies *peter*. These are independent conditions, so we can have *latin* and *peter* in any combination; and neither condition implies, or is implied by, *quads*. So we have a new hierarchy, independent of normal-algebraic-central, with the three levels (i) no requirement of a cruciform, (ii) *latin*, and (iii) *latin and peter*.

We might also consider sideways cruciforms, the so-called Nordic Cross that appears on modern Scandinavian flags and occasionally in German symbology [5]. Then, a fourth level of the hierarchy is (iv) *latin and peter and nordic*. The ordering of our hierarchy by Latin, then St. Peter, then Nordic, is of course cultural, not mathematical: It seems unlikely that Dürer would have desired the "later" cruciforms in our hierarchy before the "earlier" ones. The Latin cross was surely the most meaningful of the three to Dürer, hence first.

4 Counts of Squares

It is straightforward by computer to enumerate the squares for every choice of levels in the two hierarchies that we have defined, with and without the condition *1514AD*. Table 1 shows the results of such an enumeration. As the conditions become more restrictive along both hierarchies, the number of squares decrease, as expected. There are still 8 different squares left in the most restrictive case, 2 of which also satisfy *1514AD*.

Table 1 counts as distinct squares related by the symmetries of each class. Counting only one of each symmetry equivalence classes is a bit tricky, because the different cruciform categories have different symmetries. With no cruciform requirement, the symmetry group is D_4, so there are 8 squares in each class, related by rotations and reflections. For squares with *latin* but not *peter* or

	no cruciform required	*latin*	*latin and peter*	*latin and peter and nordic*
normal	7040 40	308 4	96 4	8 2
algebraic	3456 16	224 4	96 4	8 2
central	384 8	48 4	48 4	8 2

Table 1: Number of different magic squares with various combinations of properties. Upper value: all, lower value: only with *1514AD*. Each entry is also included in totals above it, and to its left.

	no cruciform required	*latin*	*latin and peter*	*latin and peter and nordic*
normal	880 20	129 2	23 2	1 1
algebraic	432 8	87 2	23 2	1 1
central	48 4	11 2	11 2	1 1

Table 2: Number of magic squares with various combinations of properties, counting only one member of each equivalence class under the appropriate reflection and/or rotation symmetries. The columns in the table have different symmetry properties, see text.

nordic, there is only a single left-right reflection symmetry, C_2, with 2 squares in each class. Adding *peter* gives the symmetry group of a rectangle, $D_2 = C_2 \times C_2$, with 4 squares in each class. Further adding *nordic* restores the full D_4 symmetry.

Table 2 shows the same information as Table 1, but now counting only one square in each symmetry equivalence. As an example, consider the entry "algebraic, *latin*". From Table 1 we see that, of the 224 squares in this category, 8 have D_4 symmetry, $(96-8)$ have D_2 symmetry, and $(224-96)$ have C_2 symmetry. So the number of equivalence classes is $8/8 + 88/4 + 128/2 = 87$, which is the entry in Table 2. All of the entries in Table 2 with *1514AD* required are simply half of the corresponding entries in Table 1, because the only symmetry is a C_2 reflection around a horizontal axis.

Looking at the entries in Table 2 for which *1514AD* is required (lower values in each table position), we see many values of 2. These all correspond to the pair of squares DA and DA′, equation (8). The entry with the value 4 corresponds to all of the squares in equation (8), namely DA, DA′, AD, and AD′. The entry with the value 8 (no cruciform requirement and no central requirement) adds to the above 4 squares that we have not seen before:

$$\begin{bmatrix} 13 & 2 & 3 & 16 \\ 7 & 12 & 9 & 6 \\ 10 & 5 & 8 & 11 \\ 4 & 15 & 14 & 1 \end{bmatrix} \equiv \text{DA1} \quad \begin{bmatrix} 13 & 2 & 3 & 16 \\ 11 & 8 & 5 & 10 \\ 6 & 9 & 12 & 7 \\ 4 & 15 & 14 & 1 \end{bmatrix} \equiv \text{DA2}$$
$$\begin{bmatrix} 16 & 2 & 3 & 13 \\ 7 & 9 & 12 & 6 \\ 10 & 8 & 5 & 11 \\ 1 & 15 & 14 & 4 \end{bmatrix} \equiv \text{AD1} \quad \begin{bmatrix} 16 & 2 & 3 & 13 \\ 11 & 5 & 8 & 10 \\ 6 & 12 & 9 & 7 \\ 1 & 15 & 14 & 4 \end{bmatrix} \equiv \text{AD2} \tag{9}$$

The squares in equation (9) are not central, but they share an equally strong property, termed "symmetric" by Kraitchik [4]: Pairs of number that are symmetric around the horizontal median sum to 17. The above four squares are thus easy to find, by essentially the same construction detailed in §2.4. However, we can infer that Dürer did not go down this path (or if he did, he rejected it), since his square DA is not in this set.

And what about the unique square (entries in Table 2 with the value 1) that emerges when *latin and peter and nordic* are imposed, independently of whether the square is normal, algebraic, or central? Interestingly, this is *not* Dürer's square DA. Rather it is DA′.

5 Discussion and Conclusions

It seems likely that Dürer knew about the four squares DA (his chosen square), DA′, AD, and AD′. DA′ is obtained from DA by the simple exchange of its two middle rows, while AD is obtained from DA by the exchange of its two outer columns. From either of these, AD′ is a simple exchange away. We already saw, in §2.4, that these four squares emerge together from a simple trial-and-error construction in which *sumcenter* is required.

Table 2 implies a stronger result about DA and DA′: It seems almost certain that Dürer must have known about these two squares. Absent any cruciform condition, they are two of the 4 central squares already discussed. Imposing any cruciform condition among *latin*, *peter*, or *latin and peter*, along with *1514AD* of course, yields exactly the two squares DA and DA′. Imposing *nordic* as a further condition actually eliminates DA, but leaves DA′. (And we know that Dürer knew DA!)

In other words, even without knowing exactly how Dürer constructed his square DA, we can be nearly certain that his method would have produced DA′, differing by a single row exchange, at the same time. The next question is, of course, which of the two is the "better" square?

$$\begin{bmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{bmatrix} \equiv \mathrm{DA} \qquad \begin{bmatrix} 16 & 3 & 2 & 13 \\ 9 & 6 & 7 & 12 \\ 5 & 10 & 11 & 8 \\ 4 & 15 & 14 & 1 \end{bmatrix} \equiv \mathrm{DA}' \tag{10}$$

As compared to DA, DA′ has four additional cruciform sums, the horizontal Nordic crosses. Harder to quantify aesthetically, but very visible, DA′ has columns whose elements are in a pleasingly sorted ascending or descending order. It is hard, if not impossible, to find any property of DA that is aesthetically superior to DA′.

It thus appears that in *Melencolia I*, Dürer is revealing to us only his second-best magic square. Why would he do this? It is not hard to speculate on the answer. The theme of the engraving itself furnishes the clue (Figure 1). While an unattainable perfection beckons from beyond the rainbow, the morose seated figure, an angel of uncertain gender with all the most modern scientific and mathematical tools disposed around him/her, contemplates the incompleteness of some intellectual or creative endeavor as the hourglass of time runs out. Among the various other representations of incomplete works in the engraving, the not-quite-perfect magic square DA fits in very nicely.

We want to call out to the angel, "Don't worry! Be happy! Interchange the second and third rows!"

References

[1] Wikipedia, "Melencolia I", at http://en.wikipedia.org/wiki/Melencolia_I (seen December 25, 2009).

[2] Dan Brown, 2009, *The Lost Symbol* (New York, Doubleday).

[3] Wikipedia, "Magic Square", at http://en.wikipedia.org/wiki/Magic_square (seen December 25, 2009).

[4] Maurice Kraitchik, 1942, *Mathematical Recreations*, 2nd revised ed. (reprinted New York, Dover, 1953), pp. 186–191.

[5] Wikipedia, "Nordic Cross flag", at http://en.wikipedia.org/wiki/Nordic_Cross_flag (seen December 25, 2009).

Figure 1: Albrecht Dürer's *Melencolia I* (1514)