NUMERICAL RECIPES

Webnote No. 6, Rev. 1

Derivation of the Levin Transformation

A large class of modern nonlinear transformations can be derived by using a *model sequence*

$$\tilde{s}_n = s + \tilde{r}_n \tag{1}$$

The transformation will be exact for the model sequence, and presumably will work well for other sequences with similar asymptotic properties. We partition the remainder $\tilde{r}_n$ into a remainder estimate ω_n times another quantity z_n:

$$\tilde{s}_n = s + \omega_n z_n \tag{2}$$

Here the "left-over" piece z_n is written as a finite sum of "basis functions,"

$$z_n = \sum_{j=0}^{m} c_j \phi_j(n) \tag{3}$$

where the basis functions are some asymptotic sequence, such as $1/(n+1)^j$. The key point is that you know some linear operator L that annihilates the basis functions: $L[\phi_j(n)] = 0$, that is, $L[z_n] = 0$. Then applying L to

$$z_n = \frac{\tilde{s}_n}{\omega_n} - \frac{s}{\omega_n} \tag{4}$$

gives

$$s = \frac{L[\tilde{s}_n/\omega_n]}{L[1/\omega_n]} \tag{5}$$

Note how the unknown coefficients c_j have disappeared from the final expression (5) for s. Equation (5) is now used on the actual sequence s_n instead of the model sequence $\tilde{s}_n$. The last step in devising the algorithm is to find an efficient way to evaluate (5), typically by recurrence relations.

As an illustration of this technique, we derive the *Levin transformation*. Start by multiplying equation (5.3.13) by $(n+\beta)^{k-1}$:

$$(n+\beta)^{k-1}\frac{s_n - s}{\omega_n} = \sum_{j=0}^{k-1} c_j (n+\beta)^{k-j-1} \tag{6}$$

Now note that the highest power of n on the right-hand side of (6) is n^{k-1}. Since a polynomial of degree $k-1$ in n is annihilated by the linear operator Δ^k, we use it as L in equation (6) and get

$$s = \frac{\Delta^k \left[\dfrac{(n+\beta)^{k-1} s_n}{\omega_n} \right]}{\Delta^k \left[\dfrac{(n+\beta)^{k-1}}{\omega_n} \right]} \tag{7}$$

We simplify this expression with some difference operator gymnastics: In terms of the shift operator E defined by

$$E f(n) = f(n+1) \tag{8}$$

we can write $\Delta f(n) = f(n+1) - f(n)$ as $\Delta = E - 1$. Thus by the binomial theorem

$$\Delta^k f(n) = (-1)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(n+j) \tag{9}$$

Applying this result to (7), we get the explicit form of Levin's transformation, equation (5.3.15) of the main text.