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Non-Uniform Random Variate Generation



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PREFACE

This text is about one small field on the crossroads of statistics, operations research and computer science. Statisticians need random number generators to test and compare estimators before using them in real life. In operations research, random numbers are a key component in large scale simulations. Computer scientists need randomness in program testing, game playing and comparisons of algorithms.

The applications are wide and varied. Yet all depend upon the same computer generated random numbers. Usually, the randomness demanded by an application has some built-in structure: typically, one needs more than just a sequence of independent random bits or independent uniform $[0,1]$ random variables. Some users need random variables with unusual densities, or random combinatorial objects with specific properties, or random geometric objects, or random processes with well defined dependence structures. This is precisely the subject area of the book, the study of non-uniform random variates.

The plot evolves around the expected complexity of random variate generation algorithms. We set up an idealized computational model (without overdoing it), we introduce the notion of uniformly bounded expected complexity, and we study upper and lower bounds for computational complexity. In short, a touch of computer science is added to the field. To keep everything abstract, no timings or computer programs are included.

This was a labor of love. George Marsaglia created CS690, a course on random number generation at the School of Computer Science of McGill University. The text grew from course notes for CS690, which I have taught every fall since 1977. A few ingenious pre-1977 papers on the subject (by Ahrens, Dieter, Marsaglia, Chambers, Mallows, Stuck and others) provided the early stimulus. Bruce Schmeiser's superb survey talks at various ORSA/TIMS and Winter Simulation meetings convinced me that there was enough structure in the field to warrant a separate book. This belief was reinforced when Ben Fox asked me to read a preprint of his book with Bratley and Schrage. During the preparation of the text, Ben's critical feedback was invaluable. There are many others whom I would like to thank for helping me in my understanding and suggesting interesting problems. I am particularly grateful to Richard Brent, Jo Ahrens, Uli Dieter, Brian Ripley, and to my ex-students Wendy Tse, Colleen Yuen and Amir

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