Chapter Fifteen
THE RANDOM BIT MODEL

1. THE RANDOM BIT MODEL.

1.1. Introduction.

Chapters I-XIV are based on the premises that a perfect uniform [0,1] random variate generator is available and that real numbers can be manipulated and stored. Now we drop the first of these premises and instead assume a perfect bit generator (i.e., a source capable of generating iid $\{0,1\}$ random variates $B_1, B_2, \ldots$),while still assuming that real numbers can be manipulated and stored, as before: this is for example necessary when someone gives us the probabilities p_n for discrete random variate generation. The cost of an algorithm can be measured in terms of the number of bits required to generate a random variate. This model is due to Knuth and Yao (1976) who introduced a complexity theory for nonuniform random variate generation. We will report the main ideas of Knuth and Yao in this chapter.

If random bits are used to construct random variates from scratch, then there is no hope of constructing random variates with a density in a finite amount of time. If on the other hand we are to generate a discrete random variate, then it is possible to find finite-time algorithms. Thus, we will mainly be concerned with discrete random variate generation. For continuous random variate generation, it is possible to study the relationship between the number of input bits needed per n bits of output, and to develop a complexity theory based upon this relationship. This will not be done here. See however Knuth and Yao (1976).

1.2. Some examples.

Assume first that we wish to generate a binomial random variate with parameters $n=1$ and $p \neq \frac{1}{2}$. This can be considered as the simulation of a biased coin flip, or the simulation of the occurrence of an event having probability p. If p were $\frac{1}{2}$, we could just exit with B_1. When p has binary expansion

$$p = 0.p_1p_2p_3 \cdots .$$

it suffices to generate random bits until for the first time $B_i \neq p_i$, and to return 1 if $B_i < p_i$ and 0 otherwise:

Binomial (1,p) generator

$i \leftarrow 0$
REPEAT
    $i \leftarrow i+1$
    Generate a random bit B.
UNTIL $B \neq p_i$
RETURN $X \leftarrow I_{B < p_i}$

If we define the uniform [0,1] random variate

$$U = 0.B_1B_2B_3 \cdots ,$$

then it is easy to see that this simple algorithm returns

$$I_{U \leq p} .$$

Interestingly, the probability of exiting after i bits is 2^{-i}, so that the expected number of bits needed is precisely 2, independent of p. We recognize in this example the inversion method.

The rejection method too has a nice analog. Suppose that we want to generate a random integer X where $P(X=i)=p_i$, $1 \leq i \leq n$, and that all probabilities p_i are multiples of $\frac{1}{M}$, where $2^{k-1} < M \leq 2^k$ for some integer k. Then we can consider consecutive k-tuples in the sequence $B_1, B_2, \ldots$ and set up a table with 2^k entries: M entries are used for storing integers between 1 and M, and the remaining entries are 0. If $p_i = l_i / M$, then the integer i should appear l_i times in the table. An integer 0 indicates a rejection. Now use

Rejection algorithm

REPEAT

Generate k random bits, forming the number $Z \in \{0,1, \ldots, 2^k-1\}$.

UNTIL $Z < M$

RETURN $X \leftarrow A[Z]$ (where A is the table of M integers)

In this algorithm, the expected number of bits required is k divided by the probability of immediate acceptance, i.e.

$$\frac{k}{\frac{M}{2^k}} \leq 2k = 2\left\lceil \log_2 M \right\rceil .$$

In both examples provided here, we can consider the complete unbounded binary tree in which we travel down by turning left when $B_i=0$ and right when $B_i=1$. In the rejection method, we have designated M nodes at the k-th level as terminal nodes. The remaining nodes at the k-th level are "rejection nodes", and are in turn roots of similar subtrees. Since these rejection nodes are identified with the overall root, we can superimpose them on the root, and form a pseudo-tree with some loopbacks from the k-th level to the root. But then, we have a finite directed graph, or a finite state machine.

In the inversion method, the expansion of p determines an unbounded path down the tree, and so does the expansion of U. Since we need only determine whether one path is to the left or the right of the other path, it suffices to travel down until the paths separate. With probability $\frac{1}{2}$, they separate right away. Otherwise, they separate with probability $\frac{1}{2}$ at the next level, and so forth.

What we will do in the sections that follow is

(i) Develop a lower bound for the expected number of bits in terms of $p_1, p_2, \ldots,$ the probability vector of the discrete random variate.

(ii) Develop black box methods and study their expected complexity.

2. THE KNUTH-YAO LOWER BOUND.

2.1. DDG trees.

Suppose that we wish to generate a discrete random variate X with probability vector $p_1, p_2, \ldots$. The probability vector can be finite or infinite dimensional. Every algorithm based upon random bits can be represented as a binary tree (which is usually infinite), containing nodes of two types:

(i) Branch nodes (or internal nodes), having two children. We can travel to the left child when a 0 bit is encountered, and to the right child otherwise.

(ii) Terminal nodes without children. These nodes are marked with an integer to be returned.

It is instructive to verify that this structure is present for the examples of the previous section. For example, for the binomial $(1,p)$ generator, consider the path for p, and assign terminal nodes marked 1 to all left children of nodes on the path that do not belong to the path themselves, and terminal nodes marked 0 to all right children of nodes on the path that do not belong to the path themselves.

Let us introduce the notation $t_i(k)$ for the number of terminal nodes on level k (the root is on level 0) which are marked i. Then we must have

$$\sum_{k \geq 0} \frac{t_i(k)}{2^k} = p_i \quad \text{(all } i\text{)} .$$

When these conditions are satisfied, we say that we have a DDG-tree (discrete distribution generating tree, terminology introduced by Knuth and Yao, 1976). The corresponding algorithms are called DDG-tree algorithms. DDG-tree algorithms halt with probability one because the sum of the probabilities of reaching the terminal nodes is

$$\sum_i \sum_{k \geq 0} \frac{t_i(k)}{2^k} = \sum_i p_i = 1 .$$

2.2. The lower bound.

Let us introduce the function $\chi(x) = x \bmod 1 = x - \lfloor x \rfloor$, the fractional part of x. Define furthermore

$$\nu(x) = \sum_{k \geq 0} \frac{\chi(2^k x)}{2^k} \quad (0 \leq x \leq 1) ,$$

and the entropy function

$$H(x) = x \log_2 \frac{1}{x} \quad (x > 0) .$$

Theorem 2.1

Let N be the number of random bits taken by a DDG-tree algorithm. Then:

A. $E(N) \geq \sum_i \nu(p_i)$.

B. Let $H(p_1,p_2,\ldots) = \sum_i H(p_i)$ be the entropy of the probability distribution $(p_1,p_2,\ldots)$. Then

$$H(p_1,p_2,\ldots) \leq \sum_i \nu(p_i) .$$

C. $\sum_i \nu(p_i) \leq H(p_1,p_2,\ldots)+2$.

Proof of Theorem 2.1.

We begin with an expression for $E(N)$:

$$E(N) = \sum_{k \geq 0} P(N>k)$$
$$= \sum_{k \geq 0} \frac{b(k)}{2^k}$$

where $b(k)$ is the number of internal (or: branch) nodes at level k. We obtain the lower bound by finding a lower bound for $b(k)$. Let us use the notation $t(k)$ for the number of terminal nodes at level k. Thus,

$$b(0)+t(0) = 1 ,$$
$$b(k)+t(k) = 2b(k-1) \quad (k \geq 1) .$$

Using these relations, we can show that

$$b(k) = \sum_{j>k} \frac{t(j)}{2^{j-k}} .$$

(Note that this is true for $k=0$, and use induction from there on.) But

$$\sum_{j>k} \frac{t_i(j)}{2^j} = p_i - \sum_{0 \leq j \leq k} \frac{t_i(j)}{2^j} \geq \frac{\chi(2^j p_i)}{2^j} .$$

This is true because the left-hand-sum is nonnegative, and the right-hand-sum is an integer multiple of 2^{-k}. Combining all of this yields

$$b(k) \geq \sum_i \chi(2^k p_i) .$$

This proves part A. Part B follows if we can show the following:

$$H(x) \leq \nu(x) \leq H(x)+2x \quad (\text{all } x) .$$

Note that this is more than needed, but the second part of the inequality will be useful elsewhere. For a number $x \in [0,1)$, we will use the notation $x = 0.x_1 x_2 \cdots$ for the binary expansion. By definition of $\nu(x)$,

$$\nu(x) = \sum_{k \geq 0} 2^{-k} \sum_{j > k} \frac{x_j}{2^{j-k}}$$

$$= \sum_{j \geq 0} \sum_{0 \leq k \leq j} \frac{x_j}{2^j}$$

$$= \sum_{j \geq 0} \frac{j x_j}{2^j} .$$

Now, $\nu(0) = H(0) = 0$. Also, if $2^{-k} \leq x < 2^{1-k}$,

$$\nu(x) = \sum_{j \geq k} \frac{j x_j}{2^j}$$

$$\geq \sum_{j \geq k} \frac{\log_2(\frac{1}{x}) x_j}{2^j}$$

$$= H(x) .$$

Also, because $x_k = 1$,

$$H(x) + 2x - \nu(x) = \sum_{j \geq k} \frac{(\log_2(\frac{1}{x}) + 2 - j) x_j}{2^j}$$

$$> \sum_{j \geq k} \frac{(k+1-j) x_j}{2^j}$$

$$= 2^{-k} - \sum_{j \geq k+2} \frac{(j-k-1) x_j}{2^j}$$

$$> 2^{-k} - \sum_{j \geq 1} \frac{j}{2^{j+k+1}}$$

$$= 0 . \blacksquare$$

The lower bound of Theorem 2.1 is related to the entropy of the probability vector. Let us briefly look at the entropy of some probability vectors: if $p_i = \frac{1}{n}$, $1 \leq i \leq n$, then

$$H(p_1, \ldots, p_n) = \log_2 n .$$

In fact, because H is invariant under permutations of its arguments, and is a concave function, it is true that for probability vectors

$(p_1, \ldots, p_n), (q_1, \ldots, q_n)$,

$$H(p_1, \ldots, p_n) \le H(q_1, \ldots, q_n),$$

when the p_n vector is stochastically smaller than the q_n vector, i.e. if the p_i's and q_i's are in increasing order, then

$$p_1 \le q_1 ;$$
$$p_1+p_2 \le q_1+q_2 ;$$
$$\ldots$$
$$p_1+p_2+ \cdots +p_n \le q_1+q_2+ \cdots +q_n .$$

This follows from the theory of Schur-convexity (Marshall and Olkin, 1979). In particular, for all probability vectors $(p_1, \ldots, p_n)$, we conclude that

$$0 \le H(p_1, \ldots, p_n) \le \log_2 n .$$

Both bounds are attainable. In a sense, entropy increases when the probability vector becomes smoother, more uniform. It is smallest when there is no randomness, i.e. all the probability mass is concentrated in one point. According to Theorem 2.1, we are tempted to conclude that uniform random variates are the costliest to produce. This is indeed the case if we compare optimal algorithms for distributions, and if the lower bounds can be attained for all distributions (this will be dealt with in the next sub-section). If we consider discrete distributions with n infinite, then it is possible to have $H(p_1,p_2,\ldots)=\infty$. To construct counterexamples very easily, we note that if the p_n's are $\downarrow$, then

$$H(p_1,\ldots) \ge E(\log(X))$$

where X is a random variate with the given probability vector. To see this, note that $p_n \le \frac{1}{n}$, and thus that $-p_n \log(p_n) \ge p_n \log(n)$. Thus, whenever

$$p_n \sim \frac{c}{n \log^{1+\epsilon}(n)} ,$$

as $n \to \infty$, for some $\epsilon \in (0,1]$, we have infinite entropy. The constant c may be difficult to calculate except in special cases. The following example is due to Knuth and Yao (1976):

$$p_1 = 0 ; p_n = 2^{-\lfloor \log_2(n) \rfloor - 2 \lfloor \log_2(\log_2(n)) \rfloor - 1} \quad (n \ge 2) .$$

Note that this corresponds to the case $\epsilon=1$. Thus, we note that for any DDG-tree algorithm, $E(\log(X))=\infty$ implies $E(N)=\infty$, regardless of whether the probability vector is monotone or not. The explanation is very simple: $E(\log_2(X))$ is the expected number of bits needed to store, or describe, X. If this is ∞, there is little hope of generating X requiring only $E(N)<\infty$ provided that the distribution of X is sufficiently spread out so that no bits are "redundant" (see exercises).

2.3. Exercises.

1. **The entropy.** This is about the entropy H of a probability vector $(p_1, p_2, \ldots)$. Show the following:

   A. There exists a probability vector such that $E(\log_2(X))=\infty$, yet $E(N)<\infty$. Here X is a discrete random variate with the given probability vector. Hint: clearly, the counterexample is not monotone.

   B. Is it true that when the probability vector is monotone, then $E(\log_2(X))<\infty$ implies $H(p_1, \ldots)<\infty$?

   C. Show that the p_n 's defined by

   $$p_1 = 0 \; ; \; p_n = 2^{-\lfloor \log_2(n) \rfloor - 2 \lfloor \log_2(\log_2(n)) \rfloor - 1} \qquad (n \geq 2)$$

   form a probability vector, and that its entropy is ∞.

   D. Show that if one finite probability vector is stochastically larger than another probability vector, then its entropy is at most equal to the entropy of the second probability vector.

   E. Prove that when $x \in [0,1]$ is a power of 2, we have $\nu(x)=H(x)$, and that for any $x \in [0,1]$, $\nu(x)=2^n \nu(\frac{x}{2^n})-nx$.

3. OPTIMAL AND SUBOPTIMAL DDG-TREE ALGORITHMS.

3.1. Suboptimal DDG-tree algorithms.

We know now what we can expect at best from any DDG-tree algorithm in terms of the expected number of random bits. It is another matter altogether to construct feasible DDG-tree algorithms. Some algorithms require unwieldy set-up times and/or calculations which would overshadow the contribution to the total complexity from the random bit generator. In fact, most practical DDG-tree algorithms correspond to algorithms described in chapter III. Let us quickly check what kind of DDG-tree algorithms are hidden in that chapter.

In section III.2, we introduced inversion of a uniform [0,1] random variate U. In sequential inversion, we compared U with successive partial sums of p_n 's. This corresponds to the following infinite DDG-tree: consider all the paths for the partial sums, i.e. the path for p_1, for p_1+p_2, etcetera. In case of a finite vector, we define the last cumulative sum by the binary expansion 0.111111111.... Then generate random bits until the path traveled by the random bits deviates for the first time from any of the p_n paths. If that path in question is for p_n, then return n if the last random bit was 0 (the corresponding bit on the path is 1), and return $n+1$ otherwise. This method has two problems: first, the set-up is impossible except in the following special case: all p_n 's have a finite binary expansion, and the probability vector is finite. In all other cases, the DDG-tree must be constructed as we go along.

The analysis for this DDG-tree algorithm is not very difficult. Construct (just for the analysis) the trie in which terminal nodes are put at the points where the paths for the p_n's diverge for the first time. For example, for the probability vector

$p_1 = 0.00101$
$p_2 = 0.001001$
$p_3 = 0.101101$

we have the cumulative probabilities 0.00101,0.010011,0.1111111111.... Thus, we can put terminal nodes at the positions 00, 01, and 1. It is easy to see that once the terminal nodes are reached, then on the average 2 more random bits are needed. Thus, $E(N)=2+$ expected depth of the terminal nodes in the trie defined above. In our example, this would yield $E(N)=2+\frac{1}{2}1+\frac{1}{2}2=\frac{7}{2}$. In another example, if all the p_n's are equal to 2^{-k}, $1 \le n \le 2^k$, for some integer k, then $E(N)=2+k$, which grows as $\log_2 n$. In general, we have

$$E(N) \le 2+\sum_{i=1}^{n} p_i \left\lceil \log_2(\frac{1}{p_i}) \right\rceil \le 3+H(p_1, \ldots, p_n) .$$

This follows from a simple argument. Consider the uniform [0,1] random variate U formed by the random bits of the random bit generator. Also mark the partial sums of p_i's on [0,1], so that [0,1] is partitioned into n intervals. The expected depth of a terminal node in the trie is

$$\int_0^1 D(x)\, dx$$

where $D(x)$ is the smallest nonnegative integer k such that the 2^k dyadic partition of [0,1] is such that only one of the partial sums (0 is also considered as a partial sum) falls in the same interval. The i-th partial sum "controls" an interval in which $D(x) \le \left\lceil \log_2(\frac{1}{p_i}) \right\rceil$, and the size of the interval itself is a power of 2. Thus,

$$\int_0^1 D(x)\, dx \le \sum_{i=1}^{n} p_i \left\lceil \log_2(\frac{1}{p_i}) \right\rceil ,$$

from which we derive the result shown above. We conclude that sequential search type DDG-tree algorithms are nearly optimal for all probability vectors (compare with Theorem 2.1).

The method of guide tables, and the Huffman-tree based methods are similar, with the sole exception that the probability vector is permuted in the Huffman tree case. All these methods can be translated into a DDG-tree algorithm of the type described for the sequential search method, and the performance bounds given above remain valid. In view of the lower bound of Knuth and Yao, we don't gain by using special truncation-based tricks, because truncation corresponds to search into a trie formed with equally-spaced points, and takes time proportional to $\log_2$ of the number of intervals.

Thus, it comes as no surprise that the alias method (section III.4) has an unimpressive DDG-tree analog. We can consider the following DDG-tree algorithm: first, generate a uniform $\{1, \ldots, n\}$-valued random integer (this requires on the average $\geq \log_2 n$ and $\leq 1+\log_2 n$ random bits, as we remarked above). Then, having picked a slab, we need to make one more comparison between a uniform random variate and a threshold, which takes on the average 2 comparisons by the binomial $(1,p)$ algorithm described in section XV.1. Thus,

$$2+\log_2 n \leq E(N) \leq 3+\log_2 n \ .$$

This performance grows with n, while for the optimal DDG-tree algorithms we will see that there are sequences of probability vectors for which $E(N)$ remain bounded as $n \to \infty$. In many cases, the alias algorithm does not even come close to the lower bound of Theorem 2.1.

The rejection method corresponds to the following DDG-tree: construct a DDG-tree in the obvious fashion with two types of terminal nodes, terminal nodes corresponding to a successful return (acceptance), and rejection nodes. Make the rejection nodes roots of isomorphic trees again, and continue at infinitum.

3.2. Optimal DDG-tree algorithms.

The notation of section XV.2 is inherited. We start with the following Theorem, due to Knuth and Yao (1976). It states that optimal algorithms achieving the lower bound do indeed exist.

Theorem 3.1.

Let $(p_1, p_2, \dots, p_n)$ be a discrete probability vector (where n may be infinite). Assume first that $\sum_{i=1}^{n} \nu(p_i) < \infty$. Then there exists a DDG-tree algorithm for which

$$E(N) = \sum_{i=1}^{n} \nu(p_i) .$$

In fact, the following statements are equivalent for any DDG-tree algorithm:

(i) $P(N > k)$ is minimized for all $k \geq 0$ over all DDG-tree algorithms for the given distribution.

(ii) For all $k \geq 0$ and all $1 \leq i \leq n$, there are exactly p_{ik} terminal nodes marked i on level k where p_{ik} denotes the coefficient of 2^{-k} in the binary expansion of p_i.

(iii) $E(N) = \sum_{i=1}^{n} \nu(p_i)$.

Assume next that $\sum_{i=1}^{n} \nu(p_i) = \infty$. Then, statements (i) and (ii) are equivalent.

Proof of Theorem 3.1.

We inherit the notation of the proof of Theorem 2.1. By inspecting that proof, we note that a DDG-tree algorithm attains the lower bound (if it is finite) if and only if for all i and k, we have equality in

$$\sum_{j>k} \frac{t_i(j)}{2^j} = p_i - \sum_{0 \leq j \leq k} \frac{t_i(j)}{2^j} \geq \frac{\chi(2^j p_i)}{2^j} .$$

This means that

$$\sum_{j=0}^{k} t_i(j) 2^{k-j} = \left\lfloor 2^k p_i \right\rfloor .$$

But this says simply that $t_i(k)$ is p_{ik} for all k. The number of terminal nodes at level k for integer i is 0 or 1 depending upon the value of the k-th bit in the binary expansion of p_i. To prove that such DDG-trees actually exist, define $t_i(k)$ and $t(k)$ by

$$t_i(k) = p_{ik}$$

$$t(k) = p_1(k) + \cdots + p_n(k) .$$

Thus, we certainly have

$$\sum_{k \geq 0} 2^{-k} t_i(k) = p_i$$

$$\sum_{k \geq 0} 2^{-k} t(k) = 1 .$$

A DDG-tree with these conditions exists if and only if the integers $b(k)$ defined by

$$b(0)+t(0)=1 ,$$
$$b(k)+t(k)=2b(k-1) \quad (k\geq 1)$$

are nonnegative. But the $b(k)$'s thus defined have a solution

$$b(k)=\sum_{j>k}\frac{t(j)}{2^{j-k}} .$$

Hence $b(k)\geq 0$, and such trees exist. This proves all the statements involving (iii). For the equivalence of (i) and (ii) in all cases, we note that in Theorem 2.1, we have obtained a lower bound for $b(k)$ for all k, and that the construction of the present theorem gives us a tree for which the lower bound is attained for all k. But $P(N>k)=\frac{b(k)}{2^k}$, and we are done. ■

Let us give an example of the optimal construction.

Example 3.1. (Knuth and Yao, 1976)

Consider the transcendental probabilities

$p_1=\frac{1}{\pi}$	=0.010100010111110...
$p_2=\frac{1}{e}$	=0.010111100010110...
$p_3=1-p_1-p_2$	=0.010100000101010...

The optimal tree is inherently infinite and cannot be obtained by a finite state machine (this is possible if and only if all probabilities are rational). The optimal tree has at each level between 0 and 3 terminal nodes, and can be constructed without too much trouble. Basically, all internal nodes have two children, and at each level, we put the terminal nodes to the right on that level. This usually gives an asymmetric left-heavy tree. Using the notation I for internal node, and 1,2,3 for terminal nodes for the integers 1,2,3 respectively, we can specify the optimal DDG-tree by specifying the nature of all the nodes on each level, from left to right. In the present example, this gives

Level	Nodes			
0	I			
1	I	I		
2	I	1	2	3
3	I	I		
4	I	1	2	3
5	I	2		
6	I	2		
7	I	2		
8	I	1		
9	I	I		
10	I	I	1	3
11	I	I	1	2
12	I	I	1	3
13	I	I	1	2
14	I	1	2	3
15	I	I		

■

3.3. Distribution-free inequalities for the performance of optimal DDG-tree algorithms.

We have seen that an optimal DDG-tree algorithm requires on the average

$$E(N) = \sum_{i=1}^{n} \nu(p_i)$$

random bits. By an inequality shown in Theorem 2.1, $H(x) \le \nu(x) < H(x)+2x$, $x \in [0,1]$, we see that for optimal algorithms,

$$\sum_{i=1}^{n} H(p_i) = H(p_1, \ldots, p_n)$$

$$\le E(N) \le H(p_1, \ldots, p_n)+2 .$$

Thus, the performance is roughly speaking proportional to the entropy of the distribution. In general, this quantity is not known beforehand. Often one wants a priori guarantees about the performance of the algorithm. Thus, distribution-free bounds on $E(N)$ for the optimal algorithm can be very useful. We offer:

Theorem 3.2. (Knuth and Yao, 1976)

Let $p_1, \ldots, p_n$ be a finite probability vector. Then,

$$2-2^{2-n} \leq \sum_{i=1}^{n} \nu(p_i) \leq \lceil \log_2(n) \rceil + (n-1)2^{1-\lceil \log_2(n) \rceil} .$$

Proof of Theorem 3.2.

By definition of χ and ν,

$$\chi(2^k p_1) + \cdots + \chi(2^k p_n) \leq \min(2^k, n-1)$$

for all $k \geq 0$. The $n-1$ upper bound follows by noting that the left hand side is less than n, and that it is integer valued because it can be written as

$$2^k - \lfloor 2^k p_1 \rfloor - \cdots - \lfloor 2^k p_n \rfloor .$$

Thus,

$$\sum_{i=1}^{n} \nu(p_i) = \sum_{k \geq 0} \sum_{i=1}^{n} 2^{-k} \chi(2^k p_i)$$

$$\leq \sum_{k \geq 0} 2^{-k} \min(2^k, n-1)$$

$$= \sum_{0 \leq k \leq \lfloor \log_2(n-1) \rfloor} 1 + \sum_{k > \lfloor \log_2(n-1) \rfloor} \frac{n-1}{2^k}$$

$$= 1 + \lfloor \log_2(n-1) \rfloor + \frac{n-1}{2^{\lfloor \log_2(n-1) \rfloor}} .$$

The upper bound follows when we note that $\lfloor \log_2(n-1) \rfloor = \lceil \log_2(n) \rceil - 1$. Let us now turn to the lower bound. Using the notation of the proof of Theorem 2.1, an optimal DDG-tree always has

$$\sum_{i=1}^{n} \nu(p_i) = \sum_{k \geq 1} \frac{k t(k)}{2^k}$$

$$= \sum_{k \geq 1} \frac{k(2b(k-1) - b(k))}{2^k}$$

$$= \sum_{k \geq 0} \frac{b(k)}{2^k} .$$

Since $\sum_{k \geq 0} b(k) \geq n-1$ (there are $\geq n$ terminal nodes, and thus $\geq n-1$ internal nodes), and since conditional on the latter sum being equal to s, the minimum of $\sum_{k \geq 0} \frac{b(k)}{2^k}$ is reached for $b(0) = \cdots = b(s-1) = 1$, we see that

$$\sum_{i=1}^{n} \nu(p_i) \geq 2-2^{1-s} \geq 2-2^{2-n} . \blacksquare$$

3.4. Exercises.

1. The bounds of Theorem 3.2 are best possible. By inspection of the proof, construct for each n a probability vector $p_1, \ldots, p_n$ for which the lower bound is attained. (Conclude that for this family of distributions, the expected performance of optimal DDG-tree algorithms is uniformly bounded in n.) Show that the upper bound of the theorem is attained for

$$p_i = \begin{cases} 2^{-q}\left(\frac{2^n - 2^{i-1} - 1}{2^n - 1}\right) + 2^{-q} & , 1 \le i \le 2^q + 1 - n \ , \\ 2^{-q}\left(\frac{2^n - 2^{i-1} - 1}{2^n - 1}\right) & , 2^q + 1 - n < i \le n \ , \end{cases}$$

where $q = \lceil \log_2(n) \rceil$ (Knuth and Yao, 1976).

2. Describe an optimal DDG-tree algorithm of the shape described in Example 3.1, which requires storage of the probability vector only. In other words, the tree is constructed dynamically. You can assume of course that the p_n's can be manipulated in your computer.

3. **Finite state machines.** Show that there exists a finite state machine (edges correspond to random bits, nodes to internal nodes or terminal nodes) for generating a discrete random variate X taking values in $\{1, \ldots, n\}$ if and only if all probabilities involved are rational. Give a general procedure for constructing such finite state machines from (not necessarily optimal) DDG-trees by introducing rejection nodes and feedbacks to internal nodes. For simulating one die, find a finite state machine requiring on the average $\frac{11}{3}$ random bits. Is this optimal ? For simulating the sum of two dice, find a finite state machine which requires on the average $\frac{79}{18}$ random bits. For simulating two dice (NOT the sum), find a finite state machine which requires on the average $\frac{20}{3}$ random bits. Show that all of these numbers are optimal. Note that in the last case, we do better than just simulating one die twice with the first algorithm since this would have eaten up $\frac{22}{3}$ random bits on the average (Knuth and Yao, 1976).

4. Consider the following 5-state automaton: there is a START state, two terminal states, A and B, and two other states, S1 and S2. Transitions between states occur when bits are observed. In particular, we have:

   START + 0 → S1
   START + 1 → S2
   S1 + 0 → A
   S1 + 1 → S2
   S2 + 0 → B
   S2 + 1 → START

If we start at START, and observe a perfect sequence of random bits, then what is $P(A), P(B)$? Compute the expected number of bits before halting. Finally, construct the optimal DDG-tree algorithm for this problem and find a finite-state equivalent form requiring the same expected number of bits.

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